Reference solution for MATH2020A 2015-2016.

1) Point out TRUE or FALSE without any proof for each statment

a) The line integral $\int_C \vec{F}\cdot d\vec{r}$ of the vector field $\vec{F}(x,y,z) = \langle e^x yz, e^x z + 2yz, e^x y + y^2 + 1\rangle$ along a curve C from $(0,0,0)$ to $(1,1,1)$ is $e+2$.

Ans: TRUE.

One can easily check that $\vec{F} = \nabla f$, where
$f(x,y,z) = e^x yz + y^2 z + z$
So $\int_C \vec{F}\cdot d\vec{r} = f(1,1,1) - f(0,0,0) = e+2$.

b) If two vector fields $\vec{F}$ and $\vec{G}$ in $\mathbb{R}^3$ satisfy curl $\vec{F}$ = curl $\vec{G}$, then $\vec{F} = \vec{G} + \langle a,b,c\rangle$, where a,b,c are constants.

Ans: FALSE.

curl $\vec{F}$ = curl $\vec{G}$ $\Rightarrow$ curl$(\vec{F}-\vec{G}) = 0 \Rightarrow \vec{F}-\vec{G} = \nabla f$
for some $f: \mathbb{R}^3 \to \mathbb{R}$.

c) The flux of $\vec{F} = \langle x, 0, 0\rangle$ outward through the boundary of a parallelepiped spanned by edges $\langle 1,0,0\rangle$, $\langle 1,1,3\rangle$, $\langle 0,2,0\rangle$ is 6.

Ans: TRUE.

By divergence thm,

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV = \iiint_V dV = \left|\begin{matrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{matrix}\right| = 6$$

d) If $\vec{F}(x,y,z)$ is a vector field defined on $0 < x^2+y^2+z^2 < 4$ and curl $\vec{F} = 0$ everywhere in that region, then $\oint_C \vec{F}\cdot d\vec{r} = 0$ when C is the circle $x^2+y^2=1$ in the xy-plane oriented clockwise.

Ans: TRUE.

Choose $S = \{(x,y,z) \mid x^2+y^2+z^2=1,\ z\le 0\}$ with outward normal.

Then by Stoke's thm, $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = 0$.

e) There exists a non-zero function $f(x,y,z)$ and a non-zero vector field $\vec{F}(x,y,z)$ so that $\vec{F} = \nabla f$ and $f = \operatorname{div}\vec{F}$.

<u>Ans</u>: TRUE.

Take $f = e^x + e^y + e^z$ and $\vec{F} = \langle e^x, e^y, e^z \rangle$.

2) Find the flux of the vector field $\vec{F} = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle$ outward through a circle centred at $(1,0)$ of radius $a \neq 1$.

<u>Ans</u>: If $a<1$, then $\vec{F}$ is well-defined inside the circle.

By Green's thm,

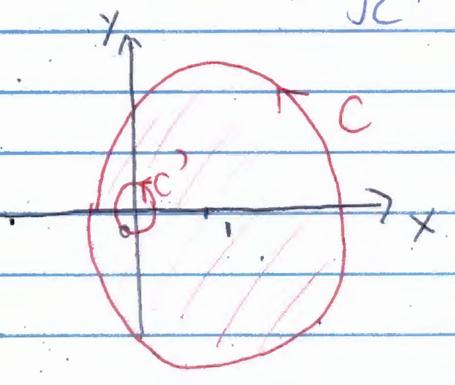
$$\oint_C \vec{F}\cdot\vec{n}\, d\gamma = \iint_D (M_x + N_y)\, dA = \iint_D \left(\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} \right) dA = 0 \,_{//}$$

If $a>1$, let C' be a sufficiently small circle centred at $(0,0)$ with radius $\varepsilon > 0$ inside C.

By Green's thm,

$$\oint_C \vec{F}\cdot\vec{n}\, d\vec{\gamma} - \oint_{C'} \vec{F}\cdot\vec{n}\, d\vec{\gamma} = \iint_D (M_x + N_y)\, dA = 0.$$

$$\Rightarrow \oint_C \vec{F}\cdot\vec{n}\, d\vec{\gamma} = \oint_{C'} \vec{F}\cdot\vec{n}\, d\vec{\gamma}$$



$$= \int_0^{2\pi} \left(\frac{\varepsilon\cos\theta}{\varepsilon^2}, \frac{\varepsilon\sin\theta}{\varepsilon^2} \right) \cdot (\varepsilon\cos\theta, \varepsilon\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} d\theta$$

$$= 2\pi \,_{//}$$

3) Consider the surface S given by the equation

$$z = (x^2+y^2+z^2)^2.$$

a) Show that S lies in the upper half space $(z \geq 0)$.

<u>Ans</u>: Clearly, $z = (x^2+y^2+z^2)^2 \geq 0$.

b) Write out the equation for the surface in spherical coordinates.

<u>Ans</u>: let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

Then $z = (x^2+y^2+z^2)^2 \Rightarrow \rho\cos\phi = \rho^4 \Rightarrow \rho^3 = \cos\phi$.

As $z \geq 0$, we have $\phi \in [0, \frac{\pi}{2}]$.
Moreover, $\theta \in [0, 2\pi]$ and $\rho \in [0, (\cos\phi)^{\frac{1}{3}}]$.

c) Find the volume of the region bounded by S.

Ans:

$$\text{Volume} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{(\cos\phi)^{\frac{1}{3}}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\cos\phi}{3} \sin\phi \, d\phi$$

$$= \frac{\pi}{3} \int_0^{\frac{\pi}{2}} \sin 2\phi \, d\phi$$

$$= \frac{\pi}{3}$$

4) Compute the surface area of the surface parametrized by $\vec{r}(u,v) = \langle u^2+v, u, v \rangle$ where $0 \leq v \leq 4$ and $\frac{v}{4} \leq u \leq 1$

Ans: $\vec{r_u} = \langle 2u, 1, 0 \rangle$, $\vec{r_v} = \langle 1, 0, 1 \rangle$.

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} i & j & k \\ 2u & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \langle 1, -2u, -1 \rangle$$

$$\Rightarrow \|\vec{r_u} \times \vec{r_v}\| = \sqrt{2+4u^2}$$

$$\Rightarrow \text{Surface Area} = \int_0^4 \int_{\frac{v}{4}}^1 \sqrt{2+4u^2} \, du \, dv$$

$$= \int_0^1 \int_0^{4u} \sqrt{2+4u^2} \, dv \, du$$

$$= \int_0^1 4u\sqrt{2+4u^2} \, du$$

$$= \frac{1}{2}\int_0^1 \sqrt{2+4u^2} \, d(2+4u^2)$$

$$= \frac{1}{3}\left(6^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$

5) Let $R = \mathbb{R}^3 \setminus \{\vec{0}\}$ and $\vec{F} = \rho^n \langle x, y, z \rangle$, where $\rho = \sqrt{x^2+y^2+z^2}$.

a) Find all integers n s.t. $\operatorname{curl} \vec{F} = 0$ in R.

Ans: By symmetry, we can only check $R_y = Q_z$.

$\because R_y = \frac{\partial}{\partial y} z(x^2+y^2+z^2)^{\frac{n}{2}} = nyz(x^2+y^2+z^2)^{\frac{n}{2}-1} = Q_z.$

$\therefore \operatorname{curl} \vec{F} = 0 \quad \forall n \in \mathbb{Z}$. (It's also true for $n=0$).

b) Whenever possible, find a potential f s.t. $\vec{F} = \nabla f$ in R.

Ans: For $n \neq -2$,

$$\text{let} \begin{cases} f_x = x\rho^n & \text{--- (1)} \\ f_y = y\rho^n & \text{--- (2)} \\ f_z = z\rho^n & \text{--- (3)} \end{cases}$$

From (1),
$$f = \int x(x^2+y^2+z^2)^{\frac{n}{2}} dx + C(y,z)$$
$$= \frac{1}{n+2}(x^2+y^2+z^2)^{\frac{n}{2}+1} + C(y,z)$$

$$\Rightarrow \quad f_y = y\rho^n + C_y$$

By (2), $\quad C_y = 0, \quad C = C(z).$

$$f = \frac{1}{n+2}(x^2+y^2+z^2)^{\frac{n}{2}+1} + C(z)$$

$$\Rightarrow \quad f_z = z\rho^n + C'$$

By (3), $\quad C' = 0, \quad C = \text{constant}.$

$\Rightarrow \quad f(x,y,z) = \frac{1}{n+2}(x^2+y^2+z^2)^{\frac{n}{2}+1}$ is a potential function for $\vec{F}$.

For $n=-2$, by similar argument, we can find the $f(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2)$ is a potential function for $\vec{F}$.

6) Find the flux of the vector field

$\vec{F}(x,y,z) = \langle x^3 z,\ y^3 z,\ 1+e^{x^2+y^2} \rangle$ through

the paraboloid $S = \{z + x^2 + y^2 = 1,\ z \geq 0\}$.

The unit normal $\hat{n}$ is oriented upwards.

Ans: Let $S_0 = \{(x,y,z) \mid x^2+y^2 \leq 1,\ z=0\}$ with normal vector $(0,0,-1)$

By divergence thm,

$$\iint_{S \cup S_0} \vec{F} \cdot \vec{n}\, dS = \iiint_V \operatorname{div} \vec{F}\, dV$$

Note that 1) $\iiint_V \operatorname{div} \vec{F}\, dV = \iiint_V 3(x^2+y^2) z\, dV$

$$= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 3r^2 z\ r\, dz\, dr\, d\theta$$

$$= \pi \int_0^1 3r^3 (1-r^2)^2\, dr$$

$$= \frac{\pi}{8}$$

2) $\iint_{S_0} \vec{F} \cdot \vec{n}\, dS = -\iint_S (1+e^{x^2+y^2})\, dS$

$$= -\int_0^{2\pi} \int_0^1 (1+e^{r^2})\, r\, dr\, d\theta$$

$$= -\pi - \pi \int_0^1 e^{r^2}\, dr^2$$

$$= -\pi e.$$

Therefore, $\iint_S \vec{F} \cdot \vec{n}\, dS = \frac{\pi}{8} + \pi e$

7) Find the line integral of $\vec{F}(x,y,z) = \langle 4z + \cos(\cos x), y^2, x+2y \rangle$ along the curve $\vec{r}(t) = \langle \cos t, 0, \sin t \rangle$ with $0 \le t \le 2\pi$.

Ans: Method I:

$$\vec{F}(x,y,z) = \langle 4z + \cos(\cos x), y^2, x+2y \rangle$$
$$= \langle 4z, 0, x+2y \rangle + \langle \cos(\cos x), y^2, 0 \rangle$$
$$= \vec{F}_1(x,y,z) + \vec{F}_2(x,y,z).$$

As $\operatorname{curl} \vec{F}_2 = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ \cos(\cos x) & y^2 & 0 \end{vmatrix} = 0$,

$$\oint_C \vec{F}_2 \cdot d\vec{r} = 0.$$

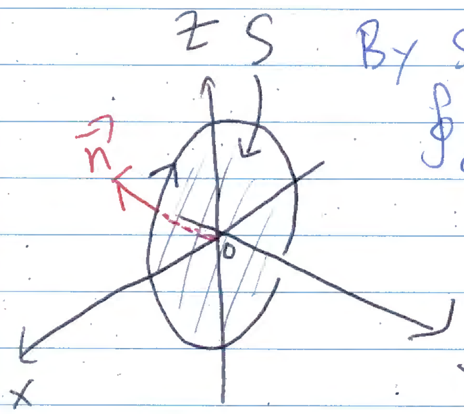
So $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F}_1 \cdot d\vec{r}$

$$= \int_0^{2\pi} \langle 4\sin t, 0, \cos t \rangle \cdot \langle -\sin t, 0, \cos t \rangle \, dt$$
$$= \int_0^{2\pi} (-4\sin^2 t + \cos^2 t) \, dt$$
$$= -3\int_0^{2\pi} \cos^2 t \, dt \quad \left(\because \int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \cos^2 t \, dt\right)$$
$$= -3\int_0^{2\pi} \frac{1+\cos 2t}{2} \, dt$$
$$= -3\pi$$

Method II:

let $S = \{(x,0,z) \mid x^2 + z^2 \le 1\}$ with unit normal $(0,-1,0)$.

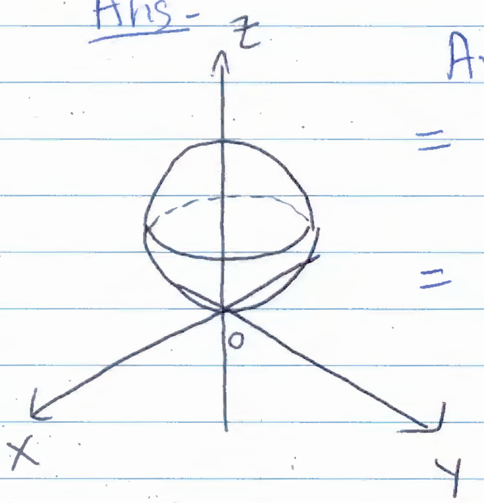
By Stoke's thm,

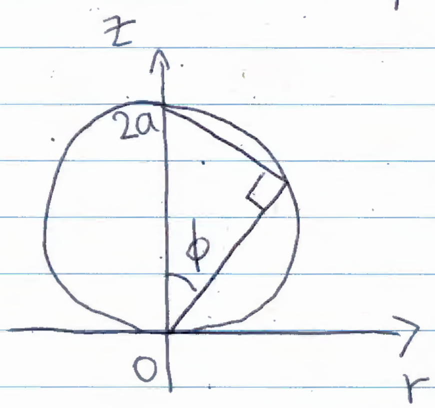


$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n} \, dS$$
$$= -3 \iint_S dS$$
$$= -3\pi$$

8) Find the average distance from a point in the solid ball of radius a to the south pole of the ball.

Ans:





Average distance

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\phi} \rho\,(\rho^2 \sin\phi)\,d\rho\,d\phi\,d\theta \Big/ \frac{4}{3}\pi a^3$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (2a\cos\phi)^4 \sin\phi\,d\phi \Big/ \frac{4}{3}\pi a^3$$

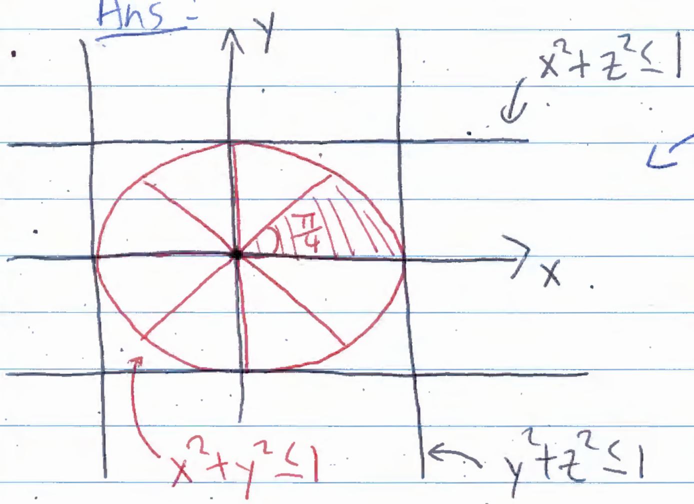
$$= 8\pi a^4\left(-\int_0^{\frac{\pi}{2}} \cos^4\phi\,d\cos\phi\right) \Big/ \frac{4}{3}\pi a^3$$

$$= \frac{8\pi a^4}{5} \Big/ \frac{4}{3}\pi a^3$$

$$= \frac{6}{5}a_{//}$$

9) Find the volume of the region which is inside all three cylinders: $x^2+y^2\le 1$, $x^2+z^2\le 1$, and $y^2+z^2\le 1$.

Ans:



From the top, the solid looks like this.
Divide the solid into 16 pieces as shown in the figure
(8 for $z\ge 0$, 8 for $z\le 0$)
In the shaded region,
$z \le \min.\{\sqrt{1-x^2}, \sqrt{1-y^2}\} = \sqrt{1-x^2}$.
By symmetry, volume of each pieces are the same.

$$\Rightarrow \text{Volume} = 16\int_0^{\frac{\pi}{4}}\int_0^1 \left(\sqrt{1-r^2\cos^2\theta}\right) r\,dr\,d\theta = \frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{1-\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{16}{3}\left(\int_0^{\frac{\pi}{4}} \sec^2\theta \, d\theta + \int_0^{\frac{\pi}{4}} \frac{1-\cos^2\theta}{\cos^2\theta} \, d\cos\theta\right)$$

$$= \frac{16}{3}\left[\tan\theta - (\cos\theta)^{-1} - \cos\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{16}{3}\left(3 - \frac{3}{\sqrt{2}}\right)$$

$$= 16 - 8\sqrt{2}$$

☆ This is the end of the course